General Certificate of Education (Adv. Level) Examination, August 2013

Combined Mathematics II - Part B

Model Answers

11.

(a) A particle is projected vertically upwards with a velocity u from a point on a fixed rigid horizontal floor. After moving under gravity it strikes the floor. The coefficient of restitution between the particle and the floor is $e(0 < e < 1)$.

(i) Sketch the velocity-time graph for the motion of the particle until the third impact.

(ii) Show that the time taken by the particle until the third impact is $\frac{2u}{g}(+e+e^2)$.

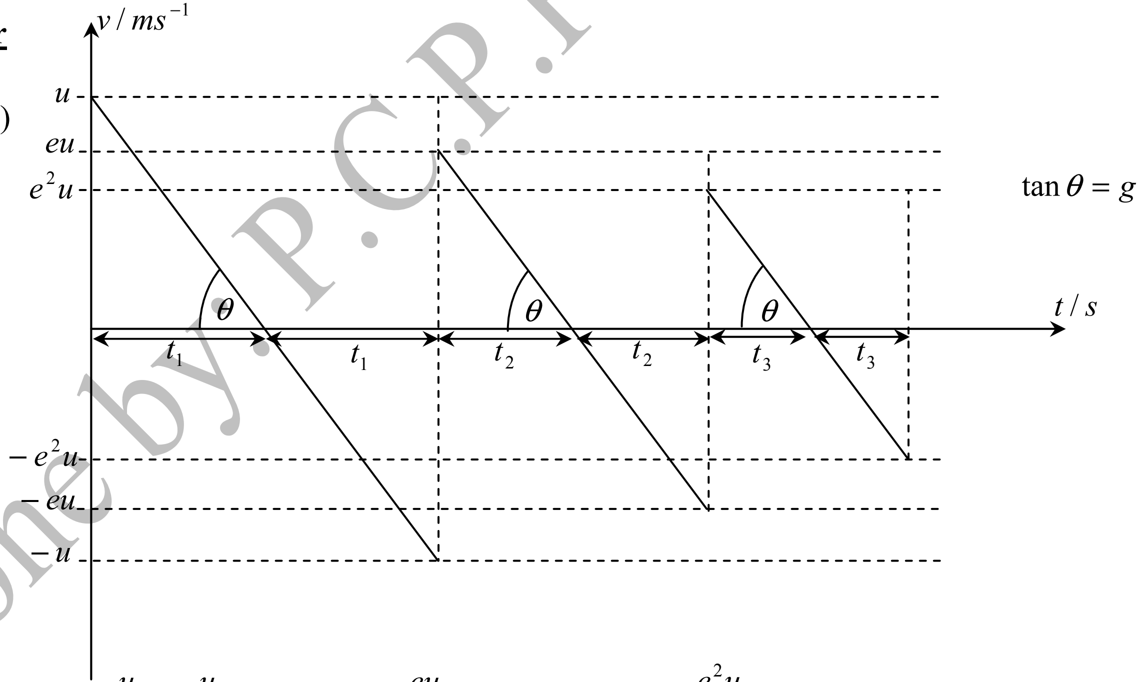
(iii) Show further that the total time taken by the particle to comes to rest is $\frac{2u}{g(1-e)}$.

(b)A train of total mass 300 metric tons moves down a straight track of inclination $\sin^{-1}\left(\frac{1}{98}\right)$ to the horizontal, at a constant speed with its engine turned off. If the magnitude of the frictional resistance for the upward motion remains at the same constant value as for the downward motion, show that the power needed to pull the train up the same track at a constant speed of $54kmh^{-1}$ is $900kW$.
Assuming that the engine is working at this power when the train is travelling on a straight horizontal track at a speed of $18kmh^{-1}$, with a resistance of same magnitude as before, find the acceleration of the train. [Take the acceleration due to gravity $g = 9.8ms^{-2}$]

Answer

(a)

(i)



(ii) $t_1 = \frac{u}{\tan\theta} = \frac{u}{g}$ $\quad t_2 = \frac{eu}{g}$ $\quad t_3 = \frac{e^2u}{g}$

Time taken by the particle until the 3rd impact $= 2t_1 + 2t_2 + 2t_3$

$$= 2\left(\frac{u}{g} + \frac{eu}{g} + \frac{e^2u}{g}\right)$$

$$= \frac{2u}{g}\left(1+e+e^2\right)$$

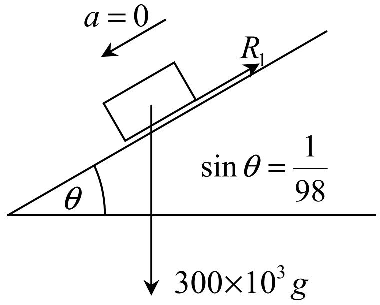
(iii) Time taken by the particle to come to rest $= \dfrac{2u}{g}\left(1+e+e^2+e^3+e^4+\ldots\right)$

$\left(1+e+e^2+e^3+e^4+\ldots\right)$ is an infinite geometric series whose first term is 1 and the common ratio e.

Since $0 < e < 1$, $\left(1+e+e^2+e^3+e^4+\ldots\right) = \dfrac{1}{1-e}$

$\therefore$ Time taken by the particle to come to rest $= \dfrac{2u}{g(1-e)}$

(b)



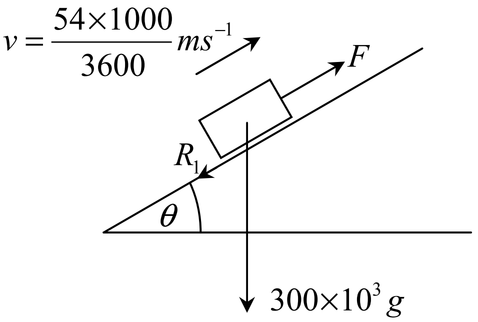
Let R_1 be the frictional resistance for the downward motion.

↙ Applying $\underline{F} = m\underline{a}$ to the train

$300\times10^3 g\sin\theta - R_1 = 0$

$R_1 = 300\times10^3\times9.8\times\dfrac{1}{98}$

$R_1 = 30000N$



Let F be the tractive force of engine of the train when the train is moving upwards along the inclined track.

↗ Applying $\underline{F} = m\underline{a}$ to the train

$F - R_1 - 300\times10^3 g\sin\theta = 300\times10^3(0)$

$F = 30000 + 300\times10^3\times9.8\times\dfrac{1}{98}$

$\therefore F = 60000N$

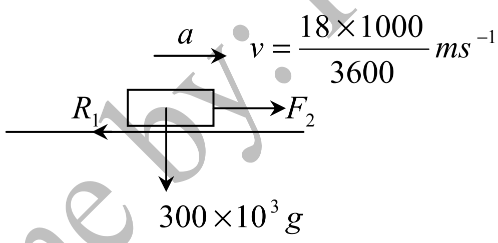
Power of the engine $=$ Tractive force $\times$ Velocity

$= 60000\times\dfrac{54\times1000}{3600}$

$= \dfrac{60000\times54\times10}{36}$ (5000, 18, 3)

$= 5\times180\times1000W$

$= \underline{\underline{900kW}}$



Let F_2 be the tractive force of the engine when the train is travelling on a straight horizontal track.

Power $= F_2\times\dfrac{18\times1000}{3600}$

$900\times10^3 = \dfrac{F_2\times18\times10^3}{3600}$ (50)

$F_2 = 180000N$

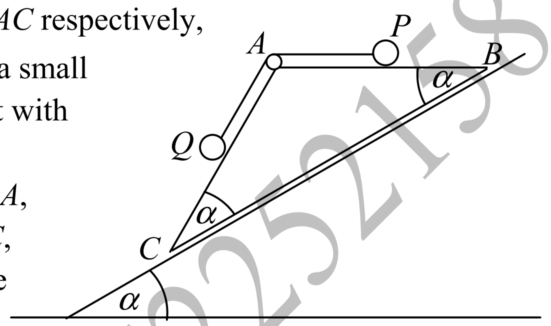
→ Applying $\underline{F} = m\underline{a}$ to the train

$180000 - 30000 = 300\times10^3 a$

$a = \dfrac{150}{300} \Rightarrow a = 0.5ms^{-2}$ $\therefore$ Acceleration of the train $= \underline{\underline{0.5ms^{-2}}}$

12.

(a) The triangle ABC is a vertical cross-section through the centre of gravity of a uniform smooth wedge of mass M. The lines AC and BC are lines of greatest slope of the respective sides, and the lines BA and AC make equal angles $\alpha\left(0<\alpha<\frac{\pi}{4}\right)$ with BC. The wedge is placed with the face containing BC on a fixed smooth plane of inclination α to the horizontal, with AB horizontal as shown in the figure. Two particles P and Q of masses m_1 and m_2 respectively, placed on AB and AC respectively, are connected by a light inextensible string which passes over a small smooth pulley at the vertex A. The system is released from rest with the string taut.



Write down the equations of motion, for the particle P along BA, for the particle Q along AC and for the whole system along BC, in order to determine the accelerations of each particle relative to the wedge and the acceleration of the wedge.

Show that if $m_1 = m_2$, the acceleration of each particle relative to the wedge is zero, and the magnitude of the acceleration of the wedge is $g\sin\alpha$.

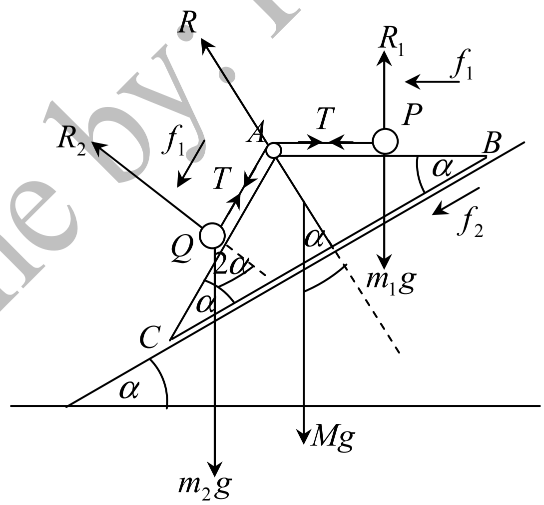
(b) A particle P of mass m is placed at the highest point of the smooth outer surface of a fixed sphere of radius a and centre O. Another particle Q of mass $2m$ moving horizontally with velocity u collides directly with P. The coefficient of restitution between P and Q is $\frac{1}{2}$. Find the velocity of P just after the collision.

Assuming that the particle P is still with the sphere when the radius OP has turned through an angle θ, show that the magnitude of the reaction on the particle P from the sphere is $\frac{m}{a}\left[ga(3\cos\theta-2)-u^2\right]$.

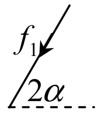
Also, show that if $u=\sqrt{ga}$, then the particle P leaves the surface of the sphere just after the collision with Q.

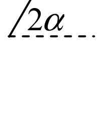
Answer

(a)

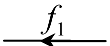
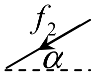


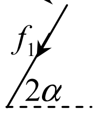
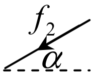
Let W- Wedge
P - Particle P
Q - Particle Q
E - Earth

$a(W,E)=$  $\quad a(P,W)=$

$a(Q,W)=$ 

From the principle of relative acceleration,

$a(P,E)=a(P,W)+a(W,E)=$  $+$ 

$a(Q,E)=a(Q,W)+a(W.E)=$  $+$ 

Applying $\underline{F} = m\underline{a}$ to the,

particle P along BA, $T = m_1(f_1 + f_2\cos\alpha)$ ------(1)

particle Q along AC, $m_2 g\sin 2\alpha - T = m_2(f_1 + f_2\cos\alpha)$ ------(2)

whole system along AC,

$m_1 g\cos(90-\alpha) + m_2 g\sin\alpha + Mg\sin\alpha = m_1(f_2 + f_1\cos\alpha) + m_2(f_2 + f_1\cos\alpha) + Mf_2$

$m_1 g\sin\alpha + m_2 g\sin\alpha + Mg\sin\alpha = m_1(f_2 + f_1\cos\alpha) + m_2(f_2 + f_1\cos\alpha) + Mf_2$ ------(3)

If $m_1 = m_2 = m$ (say)

$(1) \Rightarrow T = m(f_1 + f_2\cos\alpha)$ ------(i)

$(2) \Rightarrow mg\sin 2\alpha - T = m(f_1 + f_2\cos\alpha)$ ------(ii)

$(3) \Rightarrow 2mg\sin\alpha + Mg\sin\alpha = 2m(f_2 + f_1\cos\alpha) + Mf_2$ ------(iii)

$(i)+(ii)$;

$mg\sin 2\alpha = 2m(f_1 + f_2\cos\alpha)$

$2g\sin\alpha\cos\alpha = 2(f_1 + f_2\cos\alpha)$

$f_1 = g\sin\alpha\cos\alpha - f_2\cos\alpha$ ------(iv)

Substituting $f_1 = g\sin\alpha\cos\alpha - f_2\cos\alpha$ in (iii);

$2mg\sin\alpha + Mg\sin\alpha = 2m[f_2 + (g\sin\alpha\cos\alpha - f_2\cos\alpha)\cos\alpha] + Mf_2$

$2mg\sin\alpha + Mg\sin\alpha = 2mf_2 + 2mg\sin\alpha\cos^2\alpha - 2mf_2\cos^2\alpha + Mf_2$

$2mg\sin\alpha + Mg\sin\alpha = 2mf_2 + 2mg\sin\alpha(1-\sin^2\alpha) - 2mf_2(1-\sin^2\alpha) + Mf_2$

$2mg\sin\alpha + Mg\sin\alpha = 2mf_2 + 2mg\sin\alpha - 2mg\sin^3\alpha - 2mf_2 + 2mf_2\sin^2\alpha + Mf_2$

$Mg\sin\alpha - Mf_2 + 2mg\sin^3\alpha - 2mf_2\sin^2\alpha = 0$

$M(g\sin\alpha - f_2) + 2m\sin^2\alpha(g\sin\alpha - f_2) = 0$

$(g\sin\alpha - f_2)(M + 2m\sin^2\alpha) = 0$

Since $M > 0$, $m > 0$ and $\sin^2\alpha$, $M + 2m\sin^2\alpha \neq 0$

$\therefore\ g\sin\alpha - f_2 = 0$

$\Rightarrow f_2 = g\sin\alpha$

From (iv), $f_1 = g\sin\alpha\cos\alpha - g\sin\alpha\cos\alpha = 0$

$\therefore$ If $m_1 = m_2$, the acceleration of each particle relative to the wedge is zero, and the magnitude of the acceleration of the wedge is $g\sin\alpha$.

(b)

$e = \frac{1}{2}$

$\longrightarrow$ Applying the Principle of linear momentum

$2mu + m\times 0 = 2mv_2 + mv_1$

$v_1 + 2v_2 = 2u$ -------(1)

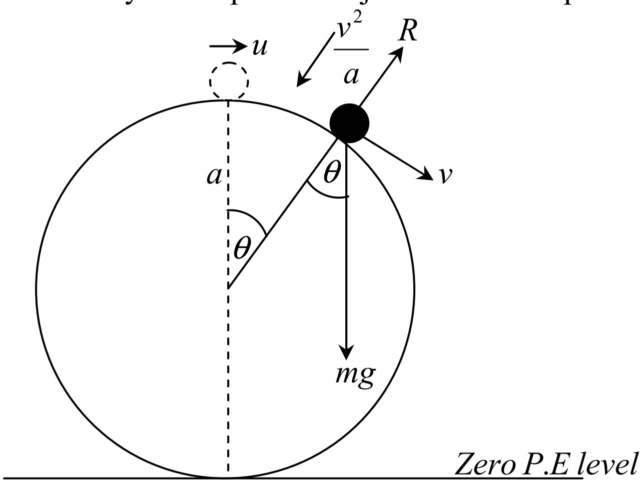
$\longrightarrow$ Applying the law of restitution

$\frac{v_1 - v_2}{0 - u} = -\frac{1}{2}$

$v_1 - v_2 = \frac{u}{2}$ ------(2)

$(1) + (2)\times 2$; $3v_1 = 3u \Rightarrow v_1 = u$

Velocity of the particle P just after the impact $= \underline{\underline{u}}$



Applying the principle of Conservation of mechanical energy

$$mg2a + \frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg(a + a\cos\theta)$$

$$4ga + u^2 = v^2 + 2ga(1 + \cos\theta)$$

$$v^2 = u^2 + 2ga - 2ga\cos\theta ----(1)$$

$\swarrow$ Applying $\underline{F} = m\underline{a}$ to the particle P

$$mg\cos\theta - R = m\frac{v^2}{a}$$

$$R = mg\cos\theta - \frac{m}{a}\left(u^2 + 2ga - 2ga\cos\theta\right)$$

$$R = \frac{m}{a}\left[ga\cos\theta - u^2 - 2ga + 2ga\cos\theta\right]$$

$$R = \frac{m}{a}\left[ga(3\cos\theta - 2) - u^2\right]$$

$\therefore$ The reaction on the particle P from the sphere $= \underline{\underline{\frac{m}{a}\left[ga(3\cos\theta - 2) - u^2\right]}}$

When the particle leaves the surface of the sphere, $R = 0$.

$\Rightarrow ga(3\cos\theta - 2) - u^2 = 0$

$\Rightarrow ga(3\cos\theta - 2) = ga \quad (\because u^2 = ga)$

$\Rightarrow 3\cos\theta = 1 + 2$

$\Rightarrow \cos\theta = 1$

$\therefore \theta = 0$

$\therefore$ If $u = \sqrt{ga}$, then the particle P leaves the surface of the sphere just after the collision with Q.

13.

A particle of mass m is attached to one end of a light elastic string of natural length l and the other end of the string is attached to a fixed point O. When the particle hangs in equilibrium the **extension** of the string is $\frac{l}{3}$. Find the modulus of elasticity of the string.

The particle is held at the point distant $\frac{l}{2}$ vertically below O, and is released from rest. Find the velocity of the particle when it first reaches the point A distant l vertically below O. Let B be the lowest point reached by the particle. Show that, for the motion of the particle from A to B, the **extension** $\boldsymbol{x}$ of the string satisfies the equation $\ddot{x}+\frac{3g}{l}\left(x-\frac{l}{3}\right)=0$.

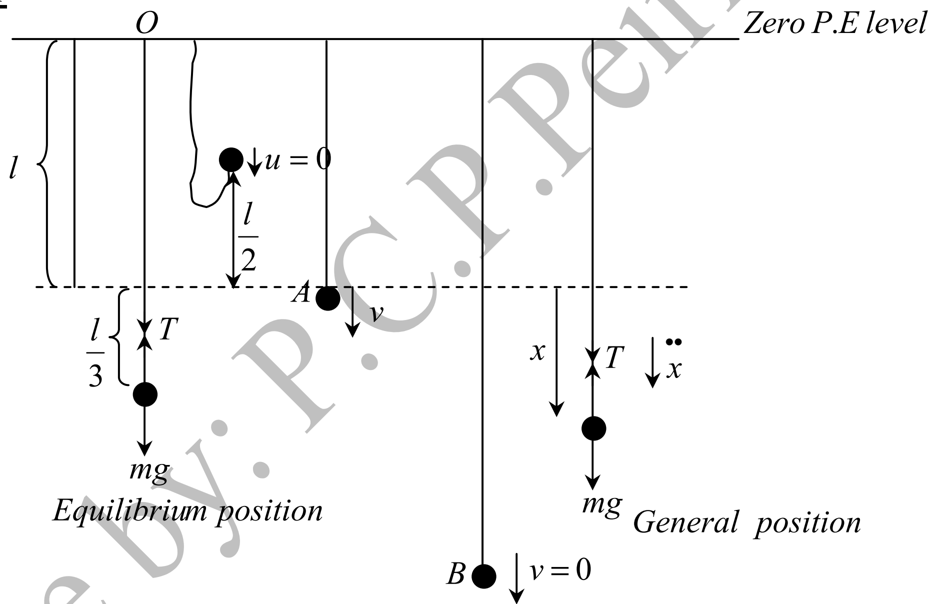
Assuming that the solution of the above equation is of the form $x=\frac{l}{3}+\alpha\cos\omega t+\beta\sin\omega t$, find the values of the constants α, β and ω.

Hence, find the centre and amplitude of the simple harmonic motion performed by the particle from A to B.

Show that the particle reaches the point B after a time $\sqrt{\frac{l}{g}}\left\{1+\frac{2\pi}{3\sqrt{3}}\right\}$ from the instant of release.

Find the tension of the string when the particle is at B.

Answer



Considering the equilibrium position of the particle,

$$T=mg$$

$$\frac{\lambda\left(\frac{l}{3}\right)}{l}=mg$$

$\lambda=3mg$; where λ is the modulus of elasticity of the string.

Applying the principle of Conservation of mechanical energy,

$$\frac{1}{2}m\times 0-mg\frac{l}{2}=\frac{1}{2}mv^2-mgl$$

$$v^2=gl$$

$$\therefore v=\sqrt{gl}$$

Velocity of the particle, when it first reaches the point $A = \sqrt{gl}$

$\downarrow$ Applying $\underline{F} = m\underline{a}$ to the particle when it at the general position

$$mg - T = m\ddot{x}$$

$$mg - 3mg\frac{x}{l} = m\ddot{x}$$

$$\ddot{x} = -\frac{3g}{l}\left(x - \frac{l}{3}\right)$$

$$\ddot{x} + \frac{3g}{l}\left(x - \frac{l}{3}\right) = 0 - - - - (*)$$

$\therefore$ The extension x of the string satisfies the equation $\ddot{x} + \frac{3g}{l}\left(x - \frac{l}{3}\right) = 0$.

$$x = \frac{l}{3} + \alpha\cos\omega t + \beta\sin\omega t - - - - (1)$$

Differentiating (1) with respect to time t

$$\dot{x} = -\alpha\omega\sin\omega t + \beta\omega\cos\omega t - - - - (2)$$

Differentiating (2) with respect to time t

$$\ddot{x} = -\alpha\omega^2\cos\omega t - \beta\omega^2\sin\omega t$$

$$\ddot{x} = -\omega^2(\alpha\cos\omega t + \beta\sin\omega t)$$

$$\ddot{x} + \omega^2\left(x - \frac{l}{3}\right) = 0 - - - - (3)$$

Comparing the equations (3) and (*), $\omega^2 = \frac{3g}{l} \Rightarrow \omega = \sqrt{\frac{3g}{l}}$

At $t = 0$, $\dot{x} = v = \sqrt{gl}$

$(2) \Rightarrow \sqrt{gl} = \beta\omega$

$$\beta = \sqrt{\frac{gl}{3g} \times l}$$

$$\therefore\ \beta = \frac{l}{\sqrt{3}}$$

At $t = 0$, $x = 0$

$(1) \Rightarrow 0 = \frac{l}{3} + \alpha$

$$\therefore\ \alpha = -\frac{l}{3}$$

When the particle is at the centre, $\ddot{x} = 0$

$(3) \Rightarrow 0 + \omega^2\left(x - \frac{l}{3}\right) = 0$

$$\therefore\ x = \frac{l}{3}$$

Thus $x = \frac{l}{3}$ is the centre of the Simple Harmonic Motion performed by the particle from A to B.

When the particle is at B, $time = t_1$, $x = x_1$ and $\dot{x} = 0$

$(2) \Rightarrow 0 = -\alpha\omega\sin\omega t_1 + \beta\omega\cos\omega t_1$

$$\alpha\omega\sin\omega t_1 = \beta\omega\cos\omega t_1$$

$\tan \omega t_1 = \frac{\beta}{\alpha} = -\frac{l}{\sqrt{3}} \times \frac{3}{l} = -\sqrt{3}$

$\therefore\ \omega t_1 = \frac{2\pi}{3}$

Substituting $\omega t_1 = \frac{2\pi}{3}$ in (1),

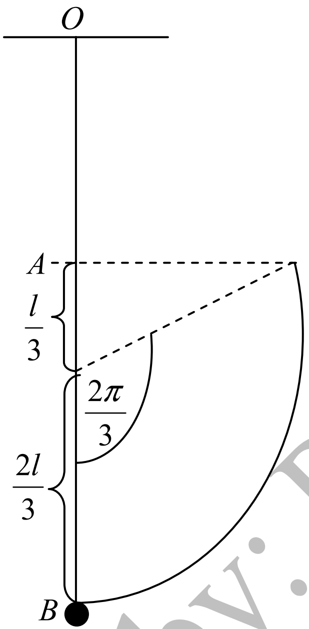
$x_1 = \frac{l}{3} - \frac{l}{3}\cos\left(\frac{2\pi}{3}\right) + \frac{l}{\sqrt{3}}\sin\left(\frac{2\pi}{3}\right)$

$x_1 = \frac{l}{3} + \frac{l}{3} \times \frac{1}{2} + \frac{l}{\sqrt{3}} \times \frac{\sqrt{3}}{2}$

$= \frac{l}{3} + \frac{l}{6} + \frac{l}{2}$

$x_1 = l$

$\therefore$ Amplitude of the S.H.M $= x_1 - \frac{l}{3} = \underline{\underline{\frac{2l}{3}}}$



Time taken by the particle to travels from A to B $= \frac{2\pi}{3\omega}$

$= \frac{2\pi}{3} \times \sqrt{\frac{l}{3g}}$

$= \sqrt{\frac{l}{g}}\,\frac{2\pi}{3\sqrt{3}}$

$\downarrow$ Applying $s = ut + \frac{1}{2}at^2$ from the initial position to A.

$\frac{l}{2} = 0 + \frac{1}{2}gt^2$

$t = \sqrt{\frac{l}{g}}$

Time taken by the particle to reaches the point B from the instant of release $= \sqrt{\frac{l}{g}} + \sqrt{\frac{l}{g}}\,\frac{2\pi}{3\sqrt{3}}$

$= \underline{\underline{\sqrt{\frac{l}{g}}\left\{1 + \frac{2\pi}{3\sqrt{3}}\right\}}}$

When the particle is at B, $T_1 = 3mg \times \frac{l}{l} \Rightarrow T_1 = \underline{\underline{3mg}}$, where T_1 is the tension in the string.

14.

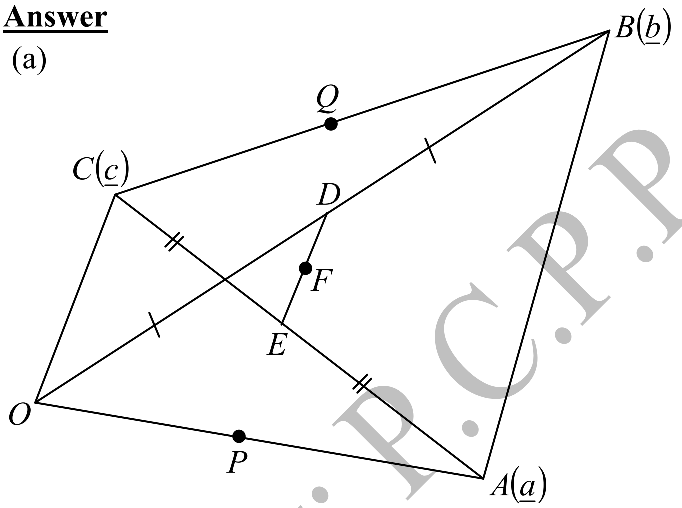
(a) Let $OABC$ be a quadrilateral, and let D and E be the mid-points of the diagonals OB and AC respectively. Also, let F be the mid-point of DE. By taking the position vectors of the points A, B and C with respect to O to be **a**, **b** and **c** respectively, show that $\overrightarrow{OF} = \frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.

Let P and Q be the mid-points of the sides OA and BC respectively. Show that the points P, F and Q are collinear and find the ratio $PF : FQ$.

(b) Let $ABCD$ be a rhombus with sides of length $2l$ and $BD = 2l$. Diagonals of the rhombus meet at the point O. Forces of magnitude $2P$, $6P$, $4P$, $8P$ and $6P$ newtons act along AB, BC, DC, DA and BD respectively, in the directions indicated by the order of the letters. Resolve the system of forces in the directions of $\overrightarrow{OC}$ and $\overrightarrow{OD}$, and show that line of action of the resultant is parallel to BC.
Also, find the moment of the system about O.
If the line of action of the resultant meets AB produced at the point E, show that $BE = 2l$.
Now, additional forces of magnitude αP, βP, γP and αP newtons are introduced to the system along EB, CE, CA and DC respectively in the directions indicated by the order of the letters. If the whole system is in equilibrium, find the values of α, β and γ.

Answer

(a)



$\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$ and $\overrightarrow{OC} = \underline{c}$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \underline{c} - \underline{a}$

Since E is the mid-point of AC, $AE = \frac{1}{2}AC$ and

direction $A \to E =$ direction $A \to C$

$\Rightarrow \overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$

$\overrightarrow{AE} = \frac{1}{2}(\underline{c} - \underline{a})$

Similarly, as D is the mid-point of OB, $\overrightarrow{OD} = \frac{1}{2}\underline{b}$

$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \underline{a} + \frac{1}{2}(\underline{c} - \underline{a})$

$\overrightarrow{OE} = \frac{1}{2}(\underline{a} + \underline{c})$

$\overrightarrow{ED} = \overrightarrow{EO} + \overrightarrow{OD} = -\frac{1}{2}(\underline{a} + \underline{c}) + \frac{1}{2}\underline{b}$

$\overrightarrow{ED} = \frac{1}{2}(\underline{b} - \underline{a} - \underline{c})$

Since F is the mid-point of ED, $\overrightarrow{EF} = \frac{1}{2}\overrightarrow{ED}$

$\overrightarrow{EF} = \frac{1}{4}(\underline{b} - \underline{a} - \underline{c})$

$\therefore \overrightarrow{OF} = \overrightarrow{OE} + \overrightarrow{EF}$

$$= \frac{1}{2}(\underline{a} + \underline{c}) + \frac{1}{4}(\underline{b} - \underline{a} - \underline{c})$$

$$\overrightarrow{OF} = \frac{1}{4}(\underline{a} + \underline{b} + \underline{c})$$

Since P and Q are the mid-points of the sides OA and BC respectively,

$$\overrightarrow{OP} = \frac{1}{2}\underline{a}$$

$$\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BQ} = \underline{b} + \frac{1}{2}\overrightarrow{BC}$$

$$= \underline{b} + \frac{1}{2}(\underline{c} - \underline{b})$$

$$\overrightarrow{OQ} = \frac{1}{2}(\underline{b} + \underline{c})$$

$$\overrightarrow{PF} = \overrightarrow{PO} + \overrightarrow{OF} = -\frac{1}{2}\underline{a} + \frac{1}{4}(\underline{a} + \underline{b} + \underline{c})$$

$$\overrightarrow{PF} = \frac{1}{4}(\underline{b} + \underline{c} - \underline{a})$$

$$\overrightarrow{FQ} = \overrightarrow{FO} + \overrightarrow{OQ} = -\frac{1}{4}(\underline{a} + \underline{b} + \underline{c}) + \frac{1}{2}(\underline{b} + \underline{c})$$

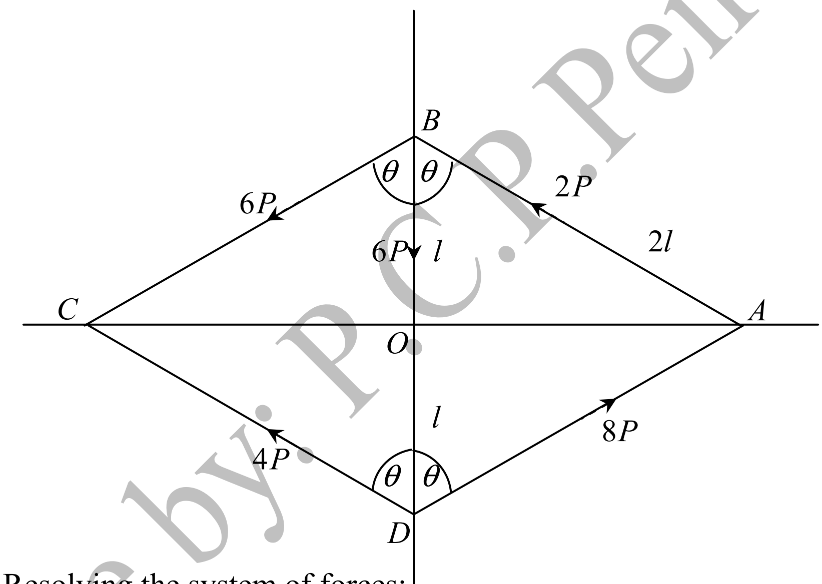
$$\overrightarrow{FQ} = \frac{1}{4}(\underline{b} + \underline{c} - \underline{a})$$

$\therefore\ \overrightarrow{FQ} = \overrightarrow{PF} \Rightarrow FQ = PF$ and FQ parallel to PF

$\therefore$ The points P, F and Q are collinear

$PF : FQ = 1:1$

(b)



$$\cos\theta = \frac{l}{2l} = \frac{1}{2}$$

$$\therefore\ \theta = 60^\circ$$

Resolving the system of forces;

in the direction of OC,

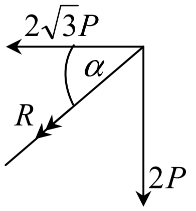
$$\overleftarrow{X} = 6P \times \frac{\sqrt{3}}{2} + 2P \times \frac{\sqrt{3}}{2} + 4P \times \frac{\sqrt{3}}{2} - 8P \times \frac{\sqrt{3}}{2}$$

$$\overleftarrow{X} = 2\sqrt{3}P$$

in the direction of OD,

$$\downarrow Y = 6P - 2P \times \frac{1}{2} + 6P \times \frac{1}{2} - 4P \times \frac{1}{2} - 8P \times \frac{1}{2}$$

$$\downarrow Y = 2P$$



Let R be the resultant and α be the angle made by the resultant with AC.

$$R^2 = (2P)^2 + (2\sqrt{3}P)^2$$

$$= (2P)^2(1+3)$$

$$\therefore\ R = 4P$$

$$\tan\alpha = \frac{2P}{2\sqrt{3}P} = \frac{1}{\sqrt{3}}$$

$$\alpha = 30^\circ$$

$\therefore$ The line of action of the resultant is parallel to BC.

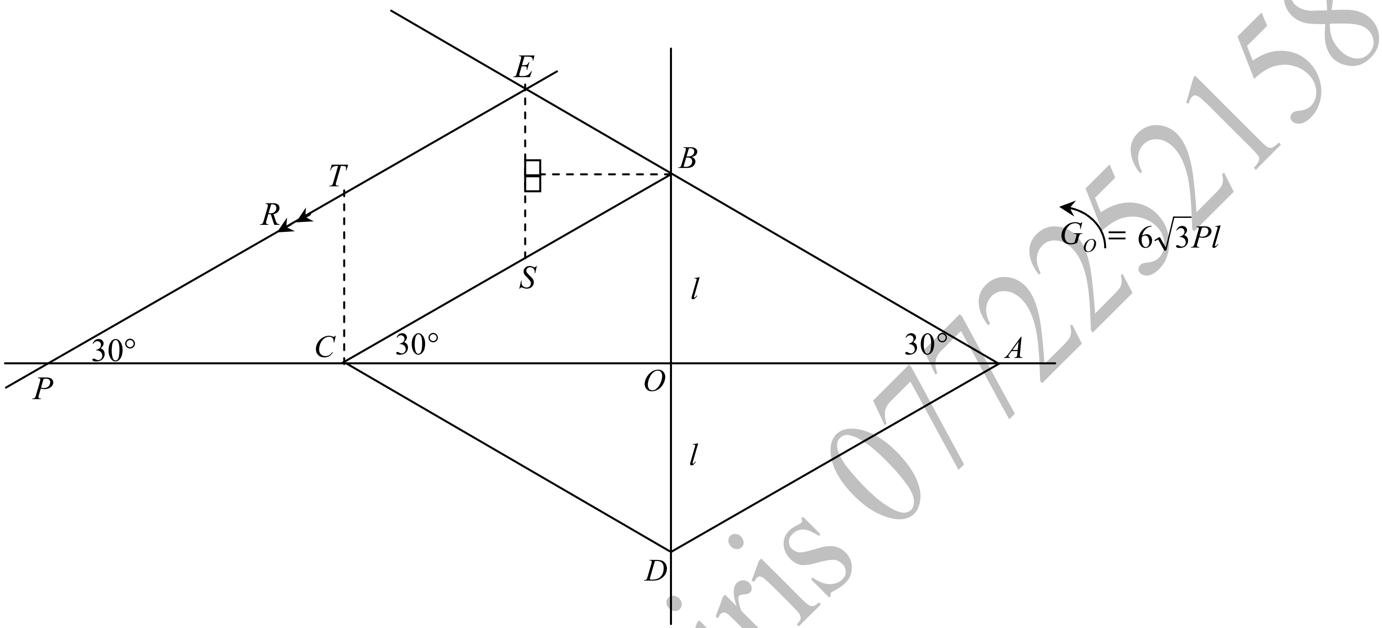
$OA = OC = 2l\sin 60^\circ = \sqrt{3}l$

Taking moments about O ↺

$$G_O = 2P\times\frac{1}{2}\times\sqrt{3}l + 6P\times\frac{\sqrt{3}}{2}\times l - 4P\times\frac{\sqrt{3}}{2}\times l + 8P\times\frac{\sqrt{3}}{2}\times l$$

$$G_O = 6\sqrt{3}Pl$$

Moment of the system of forces about $O = \underline{\underline{6\sqrt{3}Pl}}$



Let the resultant R cuts produced OC at the point P where $OP = x$.

Taking moment about O

$$\overset{\curvearrowleft}{G_O} = 6\sqrt{3}Pl$$

$$R\sin 30^\circ\times x = 6\sqrt{3}Pl$$

$$4P\times\frac{1}{2}\times x = 6\sqrt{3}Pl$$

$$\therefore\ x = 3\sqrt{3}l$$

Let S be the point of intersection of the vertical line through E and the side BC.

Let T be the point of intersection of the vertical line through C and the side PE.

$$PC = OP - OC = 3\sqrt{3}l - \sqrt{3}l$$

$$PC = 2\sqrt{3}l$$

$$CT = ES = PC\tan 30^\circ$$

$$ES = 2\sqrt{3}l\times\frac{1}{\sqrt{3}}$$

$$\therefore\ ES = 2l$$

But $ES = 2BE\sin 30^\circ = BE$

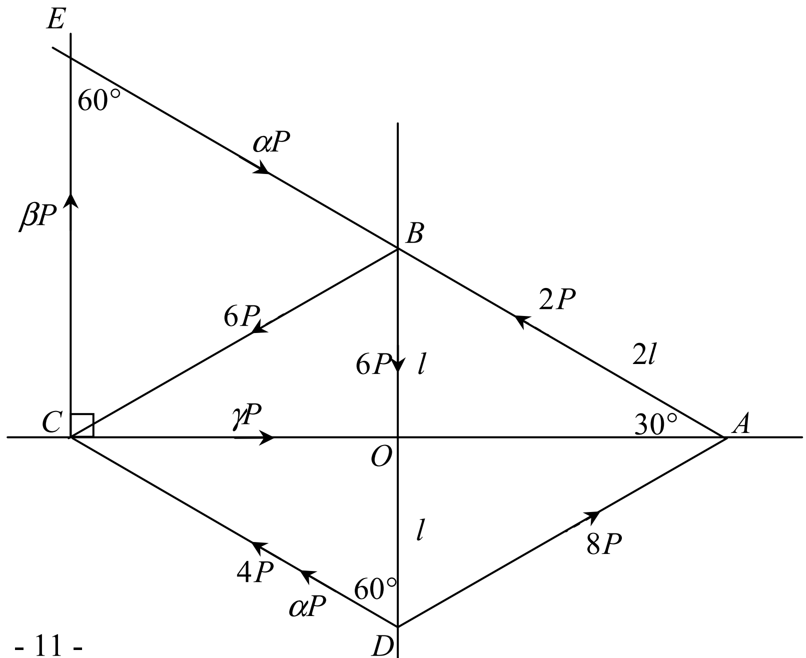
$\therefore\ \underline{\underline{BE = 2l}}$

Since the whole system is in equilibrium,

$\overset{\leftarrow}{X} = 0$, $\downarrow Y = 0$ and $\overset{\curvearrowleft}{G_O} = 0$

$$\overset{\leftarrow}{X} = 2\sqrt{3}P - \alpha P\times\frac{\sqrt{3}}{2} - \gamma P + \alpha P\times\frac{\sqrt{3}}{2}$$

$$0 = 2\sqrt{3}P - \gamma P$$



$\therefore \gamma = 2\sqrt{3}$

$\downarrow Y = 2P - \beta P + \alpha P \times \frac{1}{2} - \alpha P \times \frac{1}{2}$

$0 = 2P - \beta P$

$\therefore \beta = 2$

$\overset{\curvearrowleft}{G_O} = 0$

$6\sqrt{3}Pl - \beta P \times \sqrt{3}l - \alpha P \times \frac{\sqrt{3}}{2} \times 2l + \alpha P \times \frac{1}{2} \times \sqrt{3}l - \alpha P \times \frac{\sqrt{3}}{2} \times l = 0$

$6 - \beta - \alpha + \frac{\alpha}{2} - \frac{\alpha}{2} = 0$

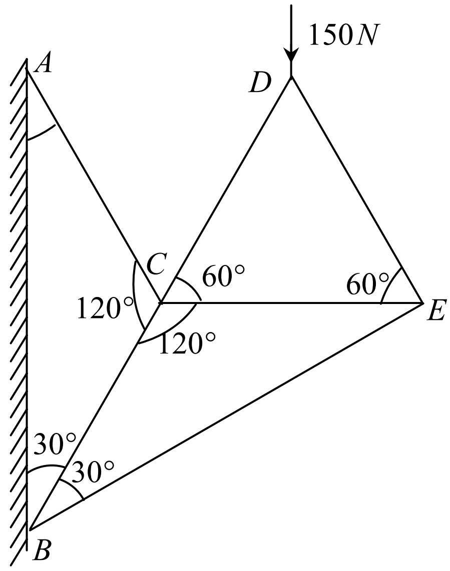
$\beta + \alpha = 6$

$\therefore \alpha = 4$

15.

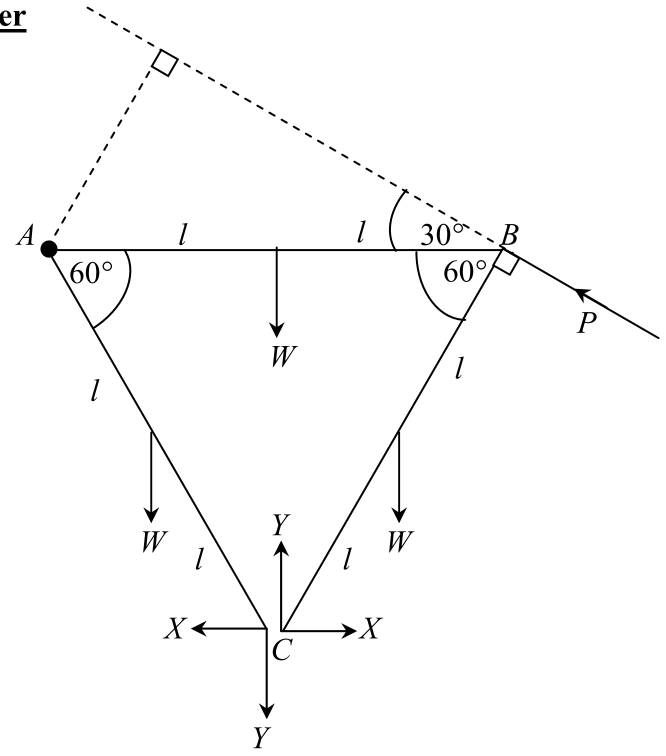
(a) Three uniform rods AB, BC and CA, each of length $2a$ and weight w are smoothly jointed at their ends to form an equilateral triangle ABC. The vertex A is smoothly hinged to a fixed point so that the triangle is free to rotate in a vertical plane. The triangle is held with AB horizontal and C below AB by a force P applied to the triangle at B perpendicular to BC in the plane of the triangle. Find the value of P. Also, find the horizontal and the vertical components of the force exerted on BC by AC at C.

(b) The adjoining figure represents a framework of six light rods smoothly jointed at the ends. It is smoothly hinged to a vertical wall at A and B, and carries a load of $150N$ at D. Draw a stress diagram using Bow's notation and **hence**, determine the stresses in the rods, indicating whether they are tensions or thrusts.



Answer

(a)



Taking moment about the point A for the system

$\overset{\curvearrowleft}{A}$;

$P \times 2l \sin 30° = W \times l + W \times l \cos 60° + W \times (l + l\cos 60°)$

$P \times 2 \times \frac{1}{2} = W + \frac{W}{2} + \frac{3W}{2}$

$\therefore \underline{\underline{P = 3W}}$

Taking moment about A for the rod AC

$\overset{\curvearrowright}{A}$;

$X \times 2l \sin 60° + Y \times 2l \cos 60° + W \times l \cos 60° = 0$

$2X \tan 60° + 2Y + W = 0$

$2\sqrt{3}X + 2Y = -W \; ---(1)$

Taking moment about B for the rod BC

$\overset{\curvearrowleft}{B}$; $-Y \times 2l \cos 60° + X \times 2l \sin 60° + W \times l \cos 60° = 0$

$-2Y + 2\sqrt{3}X = -W \; ---(2)$

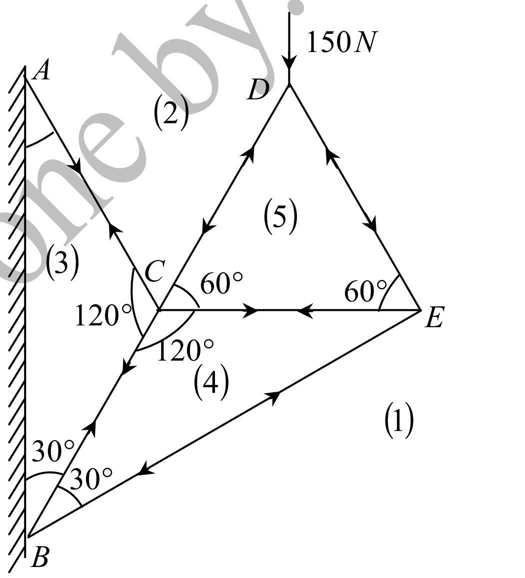
$(1)+(2)$; $4\sqrt{3}X = -2W \Rightarrow X = -\frac{W}{2\sqrt{3}}$

$(1)-(2)$; $4Y = 0 \Rightarrow Y = 0$

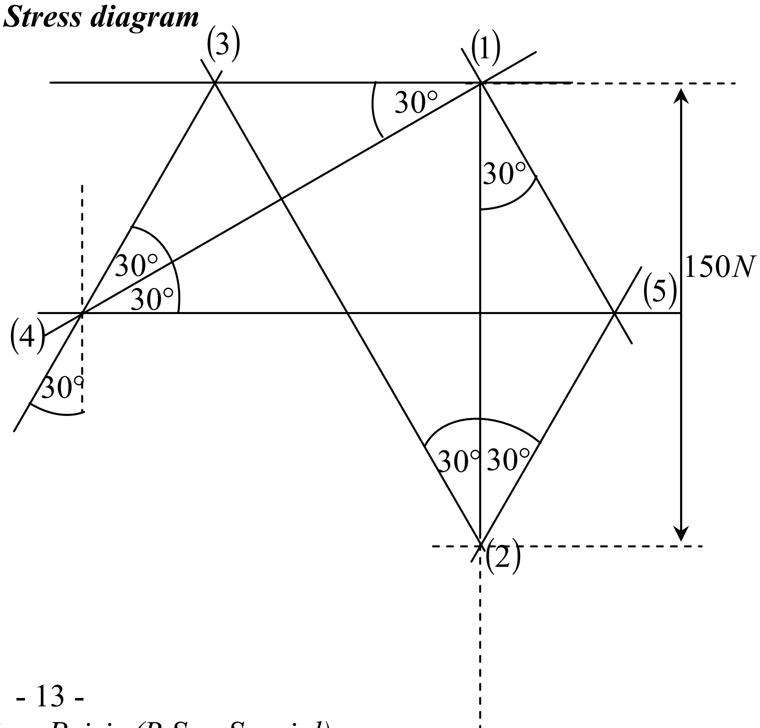
Horizontal component of the force exerted on BC by AC at $C = \underline{\underline{\frac{W}{2\sqrt{3}}}}$

Vertical component of the force exerted on BC by AC at $C = \underline{\underline{0}}$

(b)



Stress diagram

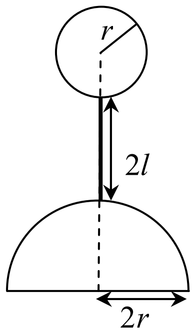


Rod	Magnitude	Nature
AC	$100\sqrt{3}N$	Tension
CD	$50\sqrt{3}N$	Thrust
CE	$100\sqrt{3}N$	Tension
DE	$50\sqrt{3}N$	Thrust
CB	$50\sqrt{3}N$	Tension
BE	$150\sqrt{3}N$	Thrust

16.

Show that the centre of mass of a uniform solid hemisphere of radius a is on its axis of symmetry, at a distance $\frac{3a}{8}$ from the centre of the base.

A composite body is made by rigidly joining a solid hemisphere and a solid sphere, made of the same uniform material, to the two ends of a uniform rod of length $2l$ and mass m in such a way that the axis of symmetry of the hemisphere, the rod and the centre of the sphere are all lying on the same line, as shown in the figure. The sphere is of radius r and mass m, and the hemisphere is of radius $2r$.

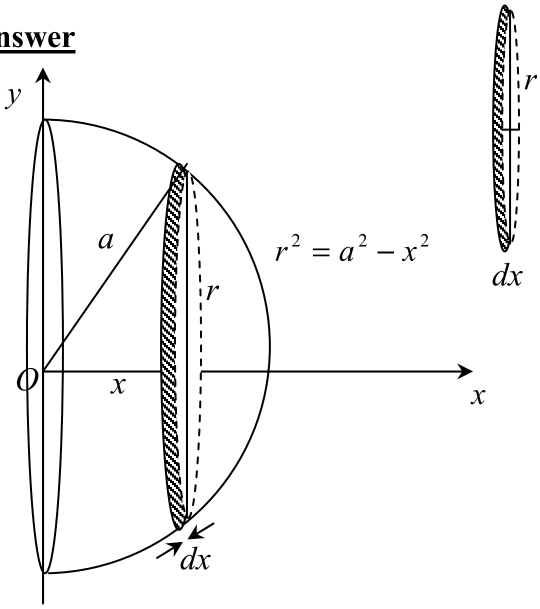


Show that the centre of mass of the composite body is at a distance $\frac{1}{6}(8r+3l)$ from the centre of the base of the hemisphere.

This composite body is placed on a fixed plane inclined at an angle θ to the horizontal with the base of the hemisphere touching the plane. Assuming that the plane is rough enough to prevent slipping, show that the composite body will not topple if $\tan\theta < \frac{12r}{8r+3l}$.

Show that if $l = \frac{4r}{3}$ and $\theta = \frac{\pi}{6}$, then the composite body will not topple and find the magnitude of the normal reaction exerted on the composite body by the inclined plane.

Answer



Since the hemisphere is symmetrical about the *x*- axis, its centre of mass should be lie on the *x*-axis.

Let $G \equiv (\bar{x}, 0)$.

Let us consider an elemental particle at a distance *x* from *O* and of thickness dx.

This elemental particle is of the shape of a cylinder of radius *r* and height *dx* where $r^2 = a^2 - x^2$.

Let ρ be the density of the hemisphere and dm be the mass of the elemental particle.

$dm = \pi r^2 dx\rho = \pi(a^2 - x^2)dx\rho$

By the definition of the centre of mass,

$$\bar{x} = \frac{\int_{x=0}^{x=a} x\,dm}{\int_{x=0}^{x=a} dm} = \frac{\int_0^a \pi x(a^2 - x^2)dx\rho}{\int_0^a \pi(a^2 - x^2)dx\rho}$$

$$= \frac{\pi\rho\int_0^a (a^2x - x^3)dx}{\pi\rho\int_0^a (a^2 - x^2)dx} = \frac{a^2\left[\frac{x^2}{2}\right]_0^a - \left[\frac{x^4}{4}\right]_0^a}{a^2[x]_0^a - \left[\frac{x^3}{3}\right]_0^a}$$

$$= \frac{\frac{a^4}{2} - \frac{a^4}{4}}{a^3 - \frac{a^3}{3}} = \frac{a^4}{4} \times \frac{3}{2a^3}$$

$$\therefore \bar{x} = \frac{3a}{8}$$

$\therefore$ The centre of mass of a uniform solid hemisphere of radius *a* is on its axis of symmetry, at a distance $\frac{3a}{8}$ from the centre of the base.

Let ρ be the density of the hemisphere and m_1 be the mass of the hemisphere.

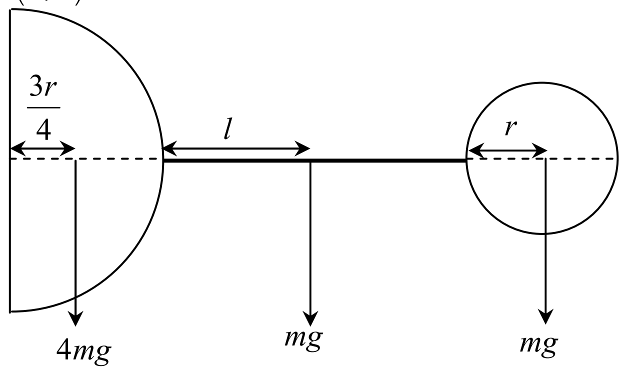
$$\rho = \frac{m}{\frac{4}{3}\pi r^3} = \frac{3m}{4\pi r^3}$$

$$\therefore m_1 = \frac{2}{3}\pi(2r)^3\rho$$

$$m_1 = \frac{2}{3}\pi \times 8r^3 \times \frac{3m}{4\pi r^3}$$

$$m_1 = 4m$$

Since the composite body is symmetrical about its axis of symmetry, its centre of gravity should lie on the axis of symmetry. Let $G \equiv (\bar{x}, 0)$



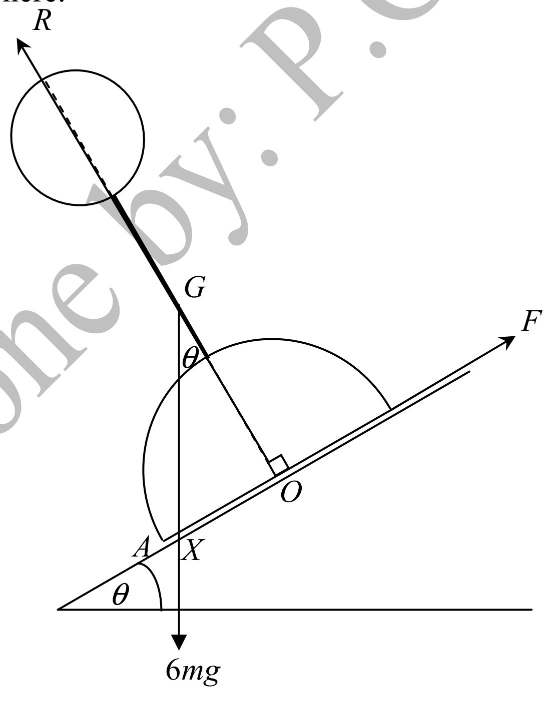
Body	Mass	Distance to the centre of mass from the base of the hemisphere
Hemisphere	$4m$	$\frac{3}{8}(2r) = \frac{3r}{4}$
Rod	m	$(2r + l)$
Sphere	m	$(3r + 2l)$
Composite body	$6m$	$\bar{x}$

Taking moment about the plane surface of the hemisphere

$$6m \times \bar{x} = 4m \times \frac{3r}{4} + m \times (2r + l) + m \times (3r + 2l)$$

$$\therefore \bar{x} = \frac{1}{6}(8r + 3l)$$

$\therefore$ The centre of mass of the composite body is at a distance $\frac{1}{6}(8r + 3l)$ from the centre of the base of the hemisphere.



$OG = \frac{1}{6}(8r + 3l)$ and $OA = 2r$

The composite body will not topple if $OX < OA$

$\Rightarrow OG \tan\theta < 2r$

$\Rightarrow \frac{1}{6}(8r + 3l)\tan\theta < 2r$

$$\Rightarrow \tan\theta < \frac{12r}{(8r + 3l)}$$

When $l = \frac{4r}{3} \Rightarrow 3l = 4r$, $\frac{12r}{8r + 3l} = \frac{12r}{8r + 4r} = 1$

When $\theta = \frac{\pi}{6}$, $\tan\theta = \frac{1}{\sqrt{3}} < 1$

If $l = \frac{4r}{3}$ and $\theta = \frac{\pi}{6}$, then the composite body will not topple.

When $l = \frac{4r}{3} \Rightarrow 3l = 4r$, $\bar{x} = \frac{12r}{6} = 2r$

Resolving the forces in the direction perpendicular to the inclined plane

$R - 6mg\cos\theta = 0$

$\Rightarrow R = 6mg\cos\dfrac{\pi}{6}$

$\therefore \underline{\underline{R = 3\sqrt{3}mg}}$

$\therefore$ The magnitude of the normal reaction exerted on the composite body by the inclined plane is $3\sqrt{3}mg$.

17.

(a) According to a survey of 100 students of a school who sat for a certain examination it was revealed that 48 students have passed the examination. Also it was revealed that out of these 100 students, 50 students have participated in sports activities in the school, 30 students have participated in musical activities in the school and none of the students have participated in both sports and musical activities. Furthermore, of those who have participated in sports activities in the school 60% have passed the examination and of those who have not participated in sports activities or musical activities in the school 30% have passed the examination.
A student is selected at random from the above 100 students.
Find the probability that this student
(i) has passed the examination given that he has participated in musical activities in the school.
(ii) has participated in sports activities in the school given that he has passed the examination.

(b) A frequency distribution of diameters of a set consisting of 50 small metal balls is given in the following table:

Diameter (cm)	Number of small balls
0.80 - 0.81	1
0.81 - 0.82	3
0.82 - 0.83	9
0.83 - 0.84	20
0.84 - 0.85	14
0.85 - 0.86	2
0.86 - 0.87	1

Calculate the first quartile of the distribution of diameters.
The mean and the standard deviation of the diameters of this set of 50 metal balls are given to be 0.835 cm and 0.01 cm respectively. Also, for another set of 100 small metal balls, it is given that the mean of the diameter is the same as that of the first set of 50 metal balls and the standard deviation is 0.015 cm.
Find the mean and the variance of the diameters of the combined set of 150 metal balls.

It is subsequently discovered that the measuring instrument used for the second set of 100 metal balls was faulty and the diameter of each ball has been underestimated by 0.015 cm. Find the true mean and true standard deviation of the diameter of these 100 metal balls.

Answer

(a) Let A = {Being a student who has passed the examination}

B_1 = {Being a student who has participated in sports activities in the school}

B_2 = {Being a student who has participated in musical activities in the school}

$P(A)=\frac{48}{100}$ $\quad P(B_1)=\frac{50}{100}$ $\quad P(B_2)=\frac{30}{100}$ $\quad P(B_1\cap B_2)=0$

$P(A/B_1)=\frac{60}{100}$ $\quad P(A/(B_1\cup B_2)^{\prime})=\frac{30}{100}$

$$P(A/(B_1\cup B_2))=1-P(A/(B_1\cup B_2)^{\prime})$$
$$=1-\frac{30}{100}=\frac{70}{100}$$

Since the events B_1 and B_2 are mutually exclusive events,

$$P(A/(B_1\cup B_2))=P(A/B_1)+P(A/B_2)$$
$$\Rightarrow \frac{70}{100}=\frac{60}{100}+P(A/B_2)$$
$$\therefore\ P(A/B_2)=\frac{10}{100}$$

$\therefore$ Probability that this student has passed the examination given that he has participated in musical activities in the school is 10%.

From the total probability theorem,

$$P(A)=P(B_1)P(A/B_1)+P(B_2)P(A/B_2)$$
$$=\frac{50}{100}\times\frac{60}{100}+\frac{30}{100}\times\frac{10}{100}$$
$$P(A)=\frac{33}{100}$$

From the conditional probability theorem,

$$P(B_1/A)=\frac{P(B_1\cap A)}{P(A)}$$
$$=\frac{P(B_1)P(A/B_1)}{P(A)}$$
$$=\frac{50}{100}\times\frac{60}{100}\times\frac{100}{33}$$
$$P(B_1/A)=\frac{10}{11}$$

$\therefore$ Probability that this student has participated in sports activities in the school given that he has passed the examination is $\frac{10}{11}$.

(b)

Diameter (cm)	Number of small balls	Cumulative frequency $F>$
0.80 - 0.81	1	1
0.81 - 0.82	3	4
0.82 - 0.83	9	13
0.83 - 0.84	20	33
0.84 - 0.85	14	47
0.85 - 0.86	2	49
0.86 - 0.87	1	50

$Q_1 = b_{Q_1} + \dfrac{c_{Q_1}\left(\dfrac{N}{4} - F_{Q_1-1}\right)}{f_{Q_1}}$, where $b_{Q_1} = 0.82$, $\dfrac{N}{4} = 12.5$, $c_{Q_1} = 0.01$, $F_{Q_1-1} = 4$ and $f_{Q_1} = 9$

$$Q_1 = 0.82 + \frac{0.01(12.5-4)}{9}$$
$$Q_1 = 0.82 + 0.01 \times \frac{8.5}{9}$$
$$Q_1 = 0.82 + 0.01 \times 0.94$$
$$Q_1 = 0.82 + 0.0094$$
$$\therefore \underline{\underline{Q_1 = 0.8294}} \text{ cm}$$

Let x be the diameter of the metal ball of the first set and y be the diameter of the metal ball of the second set.

$n = 50 \qquad \bar{x} = 0.835 \qquad \sigma_x^{\ 2} = 0.01^2$

$m = 100 \qquad \bar{y} = 0.835 \qquad \sigma_y^{\ 2} = 0.015^2$

Let $\bar{z}$ and $\sigma_z^{\ 2}$ are the mean and the variance of the combined set of 150 small metal balls respectively.

$$\bar{z} = \frac{n\bar{x} + m\bar{y}}{n+m}$$
$$= \frac{50 \times 0.835 + 100 \times 0.835}{150}$$
$$= \frac{150 \times 0.835}{150}$$
$$\underline{\underline{\bar{z} = 0.835}} \text{ cm}$$

$$\sigma_z^{\ 2} = \frac{1}{n+m}\left\{n\sigma_x^{\ 2} + m\sigma_y^{\ 2} + \frac{nm}{n+m}(\bar{x}-\bar{y})^2\right\}$$
$$\sigma_z^{\ 2} = \frac{1}{150}\left\{50 \times 1 \times 10^{-4} + 100 \times 225 \times 10^{-6} + \frac{50 \times 100}{150}(0.835 - 0.835)^2\right\}$$

$= \frac{1}{3}\{10^{-4} + 4.5\times10^{-4}\}$

$= \frac{5.5\times10^{-4}}{3}$

$\sigma_z^{\ 2} = 1.83\times10^{-4}\ \text{cm}^2$

The mean of the combined set of 150 metal balls $= 0.835cm$

The variance of the combined set of 150 metal balls $= 1.83\times10^{-4}\ cm^2$

Let y be the diameter of the metal ball of the second set.
Let l be the true diameter of the metal ball of the second set.

$l = y + 0.015$

$\therefore\ \bar{l} = \bar{y} + 0.015$ and $\sigma_l = \sigma_y$

$\therefore$ The true mean of the diameter of these 100 metal balls $= 0.835 + 0.015 = \underline{\underline{0.85cm}}$

True standard deviation of the diameter of these 100 metal balls $= \underline{\underline{0.015cm}}$